

# Coordinate independent approach to $5d$ black holes

V.E. Didenko

*I.E. Tamm Department of Theoretical Physics, Lebedev Physical Institute,  
Leninsky prospect 53, 119991, Moscow, Russia*

didenko@lpi.ru

## Abstract

Five-dimensional generalization of  $(A)dS_5$ -Kerr black hole is shown to be generated in a coordinate free way by a single  $AdS_5$  global symmetry parameter. Its mass and angular momenta are associated with Casimir invariants of the background space-time symmetry parameter leading to the black hole classification scheme similar to that of relativistic fields resulting apart from ordinary black hole to “tachyonic” and “light-like” ones.

## 1 Introduction

The generalization of Myers-Perry black holes [1] to include a non-zero cosmological constant found in [2] has provided a great deal of interest in higher-dimensional black holes. There were several attempts since then to generalize the result of [2] to add NUT-charge if not electro-magnetic charges to the metric altogether. Little progress in the generalization of higher-dimensional Einstein-Maxwell black holes has been still achieved. The authors of [3] were able, however, to add the NUT-charge yielding Kerr-NUT-(A)dS metric – the most general higher-dimensional Einstein black hole solution available to the moment. This solution in many respects resembles its four dimensional counterpart represented by Carter-Plebański metric [4, 5]. Particularly, many “mysteries” attributed to  $4d$  black holes such as complete integrability of geodesic equations and variable separation for Hamilton-Jacobi, Klein-Gordon and Dirac equations in the black hole background turned out to be resided in higher-dimensional generalization as well [6, 7, 8]. The origin of these mysteries in four dimensions was clarified by Floyd and Penrose [9] as they found a Killing-Yano tensor [10] responsible for all these miracles put together. In higher-dimensions a single Killing-Yano tensor is insufficient to provide complete integrability of the aforementioned equations. It has been shown recently in [11] that the class of Kerr-NUT-(A)dS metrics admits the so called principal conformal Killing-Yano tensor (CYK) that generates a tower of Killing-Yano tensors necessary for integrability and variable separation. Of this CYK field the authors [11] referred to as of higher-dimensional black hole hidden symmetry.

In the present paper we wish to lay the views on black holes based on unfolded formulation of dynamical systems [12, 13] which being coordinate independent in principal allows us to

identify structures that may be hidden in a particular coordinate system. For example, the unfolded approach applied to four dimensional black holes in [14] happened to be very efficient demonstrating that the  $(A)dS_4$  black hole is generated by a single  $(A)dS_4$  global symmetry parameter encoding the black hole mass and angular momentum in its two Casimir invariants. Our primary interest is higher-spin generalization of general relativity black holes. This is where the unfolded approach reveals its strength. It was shown in [15, 14] that apart from  $4d$  black hole solution in gravity sector, the  $(A)dS_4$  global symmetry parameter generates a tower of Kerr-Schild type solutions of spin  $s$  Fronsdal equations. That Kerr-Schild field *i.e.*, shear free geodesic null congruence, generates solutions of massless field equations has been known ever since Penrose transform was invented [16]. The main result of [15, 14], however, is that it is the  $AdS_4$  global symmetry parameter that determines the black hole null geodesic congruence. This is what really important for higher-spin generalization. To obtain a four dimensional higher-spin black hole, the global higher-spin symmetry parameter rather than the  $AdS_4$  one was chosen in [17]. Via an analogue of Penrose transform it has produced then a solution [17] of the  $4d$  nonlinear higher-spin equations [18, 19] which boils down to  $AdS_4$ -Schwarzschild black hole in gravity sector in the weak field limit when higher-spin fields decouple.

We would like to stress that despite often quoted, the notion of higher-spin black hole is yet to be justified. Space-time geometry in higher-spin field theories is far from being understood. Hence, the role of the metric in such theories is rather hazy not to mention that the line element associated with it is generically no longer gauge invariant [17]<sup>1</sup>. Thus, an event horizon attributed to black holes as we know them is not well defined in higher-spin gauge theories. To the moment, we prefer to take a higher-spin black hole as a solution of higher-spin field equations that has similar space-time symmetry as ordinary black hole has. Examples of such solutions can be found in [17, 21, 22, 23, 24]. An interesting conjecture on what actually should be called a higher-spin black hole in three dimensions is given in [22] and elaborated further on in [23, 24]. The authors of [22] have found a static solution of  $sl(3) \oplus sl(3)$  Chern-Simons gravity that carries spin two and spin three charges and conjectured that the eigenvalues of Chern-Simons black hole connection holonomies around the Euclidian time circle should be equal to those of the BTZ black hole [25]. Interestingly, this proposal turns out to be consistent with the integrability condition coming out from the thermodynamical partition function. Another curious result on black holes in higher-derivative theories was obtained in [26], which shows that higher-derivative terms may smooth out curvature singularities.

Aiming higher-spin generalization of black holes in  $d > 4$  we would like to understand to what extent the results of the four dimensional analysis carried out in [14] can be extended beyond  $d = 4$ . Particularly, it is of interest to see if the Kerr-Schild ansatz works for higher-spin massless fields in  $d > 4$ . On one hand, the generalization of Kerr-Schild fields for higher spins does not seem feasible as the relation between null shear free geodesic congruence and massless equations is essentially four dimensional in nature being based on the Penrose twistor transform<sup>2</sup>. On the other hand, both  $d$ -dimensional Myers-Perry example [1] and its recent generalization for a nonzero cosmological constant [2] illustrate that Kerr-Schild ansatz, for some reason, remains valid for gravity fields in arbitrary  $d$ . The other observation in favor of higher-dimensional generalization of the  $4d$  description proposed in [14] is black hole “hair” counting.

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<sup>1</sup>It should be noted that in some cases it is possible to construct Lorentz invariant line element in higher-spin gravity as has been shown for  $sl(3) \oplus sl(3)$  Chern-Simons theory in [20]

<sup>2</sup>In higher-spin theory an analogue of Penrose transform relates the adjoint module to the twisted-adjoint one in any dimension being identical to that of Penrose in  $d = 4$  (see [27] for application in  $4d$ ).

Indeed, one of the reason why a  $4d$  black hole could be generated by an AdS/Minkowski symmetry parameter is that the rank of the corresponding isometry algebra  $o(3, 2)$  or  $iso(3, 1)$  is equal to the number of black hole parameters – mass and angular momentum, *i.e.* 2. In higher dimensions, there are  $\lfloor \frac{d-1}{2} \rfloor$  angular momenta and one mass parameter, altogether  $\lfloor \frac{d+1}{2} \rfloor = \text{rank}(o(d-1, 2), o(d, 1), iso(d-1, 1))$ .

The purpose of this paper is to analyze if a general relativity  $5d$  black hole can be generated by background space-time ( $(A)dS_5$  or Minkowski) global symmetry parameter. We answer to this question in the affirmative and provide a coordinate free black hole classification in the spirit of relativistic fields classification. There are three Casimir invariants associated with the symmetry parameter – one is  $P^2$ , the other two  $I_1, I_2$  are associated with spins. These three identify the resulting black hole solution in the following way. For  $P^2 < 0$  we reproduce ordinary Myers-Perry type  $5d$  black hole originally obtained in [28] which we refer to as of the Kerr black hole case. Its two angular momenta are associated with the spin Casimir invariants  $I_1, I_2$ . For  $P^2 = 0$  the solution is called “light-like” Kerr since in Newton’s limit the “source” is located on a light-like surface rather than on a time-like one as in the case of ordinary Kerr<sup>3</sup>. For  $P^2 > 0$  the metric corresponds to “tachyonic” Kerr. Arbitrary  $P^2$  can be identified with the  $5d$  analogue of the Carter-Plebański parameter  $\epsilon$ . Just as in the four dimensional case, it can be set to be discrete  $P^2 = -1, 0, +1$ . Then, we will show how the background global symmetry condition can be deformed in such a way that its integrability condition  $d^2 = 0$  gets consistent for the black hole curvature tensor. The deformation parameter is associated with the black hole mass. The resulting system has the unfolded form analogous to that obtained in [14] and can be reduced to the initial nondeformed one by local field redefinition.

A great deal of computational simplification in classical  $4d$  general relativity results from an extensive use of two-component Weyl spinors. Particularly, Petrov classification of the Weyl tensor looks especially simple in terms of spinors. In the generic  $d$ -dimensional space-time spinor-to-vector isomorphism is hardly applicable because of the exponential growth of Clifford algebra dimension compared to vector’s linear growth.  $d = 5$  is still small enough to take advantage of the spinor approach. Particularly, the  $5d$  analogue of Petrov classification for the Weyl tensor has been recently established [29].<sup>4</sup> In deriving our results we use  $5d$  spinor formalism.

The paper is organized as follows. In section 2 we give the notation we use throughout the paper, and then recall some generalities on the Kerr-Schild ansatz for solving Einstein equations and introduce the global symmetry condition on  $(A)dS$  and Minkowski space-time. In section 3, the spinor form of this condition is studied, Casimir invariants are constructed. In subsection 3.1, we build Kerr-Schild vectors and associate massless fields on  $(A)dS_5$  background. In section 4, we generate black hole solutions and classify those on Minkowski background according to the values of Poincaré invariants. In subsection 4.2, the unfolded form of the black hole system as a deformation of the background global symmetry condition is given. Its properties and its relation to nondeformed equations are elaborated. Conclusion is given in section 5. To be self-contained, we provide four appendices. Two of them devoted to Cartan formalism and five-dimensional spinors, while in the remaining two the proof of the geodesity condition is

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<sup>3</sup>Let us stress that the notion of source is considered here in Newton’s limit and should not be confused with the true singularity of a black hole metric which, as well known from the Penrose diagram, belongs to the space-like world-line.

<sup>4</sup>As shown in [30], De Smet classification [29] in fact does not account for the reality condition. Hence, some of the cases were not possible. In [30], this gap has been filled.

given and some useful identities are summarized.

## 2 Generalities

### 2.1 Notation

In this paper, the following notation has been adopted. Latin indices from the middle of alphabet  $m, n, \dots$  are attributed to space-time tensors and range  $0, \dots, d-1$ . Latin indices from the beginning of alphabet  $a, b, \dots$  are fiber ones and range  $a, b = 0, \dots, d-1$ , raised and lowered using mostly plus flat Minkowski metric  $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$ . The background covariant derivative is denoted by  $D$ , while  $\mathcal{D}$  is the black hole one. Finally,  $5d$  spinor indices are Greek  $\alpha, \beta, \dots$  and range four values  $1, \dots, 4$ .

### 2.2 Kerr-Schild metric

It is well known since [1] that  $d$ -dimensional Einstein black holes with spherical horizon topology admit Kerr-Schild representation even in the presence of a cosmological constant [2]. This means that the black hole metric being an exact solution of Einstein equations has a perturbative form around the flat or  $(A)dS_d$  background

$$g_{mn} = \bar{g}_{mn} + \frac{2M}{H} k_m k_n, \quad g^{mn} = \bar{g}^{mn} - \frac{2M}{H} k^m k^n, \quad (2.1)$$

where  $\bar{g}_{mn}$  is the fiducial metric,  $M$  is the black hole mass,  $H$  is some function. The fluctuational part has a specific factorized form with the Kerr-Schild vector  $k^m$  being null and geodesic

$$k^m k_m = 0, \quad k^m \mathcal{D}_m k_n = 0, \quad (2.2)$$

where  $\mathcal{D}$  is a covariant with respect to the  $g_{mn}$  derivative. Indices in (2.1) and (2.2) can be raised and lowered by either background or full metric since  $k^m$  is light-like. This makes  $k^m$  a vector with respect to both metrics. Moreover, it can be verified to be geodesic for both metrics as well

$$k^m \mathcal{D}_m k_n = k^m D_m k_n = 0. \quad (2.3)$$

The detailed analysis of  $d$ -dimensional Kerr-Schild ansatz can be found in [31]. The key property of the Kerr-Schild construction is that it renders Einstein equations linear. In other words, if  $\bar{g}_{mn}$  is the  $(A)dS_d$  metric such that its Ricci tensor  $\bar{R}_{mn}(\bar{g}) = (d-1)\Lambda\bar{g}_{mn}$ , then Einstein equations for (2.1)  $R_{mn}(g) = (d-1)\Lambda g_{mn}$  reduce to the first order background free field equations

$$\square h_{mn} - D_p D_m h^p_n - D_p D_n h^p_m = -2(d-1)\Lambda h_{mn}, \quad h_{mn} = \frac{1}{H} k_m k_n \quad (2.4)$$

The nonlinear  $O(M^2)$  part is satisfied on account of (2.2) and (2.4) leaving no new constraints for  $k^m$  or  $H$ .

Finally, the last comment on Kerr-Schild is in order. It was already mentioned that  $k^m$  is a Kerr-Schild vector for both metrics. For Myers-Perry black holes, the function  $H$  is a scalar with respect to both metrics. Altogether this implies that the metric (2.1) is given in a background covariant form. As will be shown, a sufficient ingredient which generates on-shell metrics in the form (3.22) is a background space-time global symmetry parameter.

## 2.3 Background symmetries

To make use of the background space-time global symmetry parameter in the black hole description, it is convenient to work in the Cartan formalism (see appendix A). Let  $\mathbf{w}_{ab} = -\mathbf{w}_{ba} = w_{ab,n}dx^n$  be the Lorentz connection 1-form and  $\mathbf{e}_a = e_{a,n}dx^n$  vielbein 1-form.  $(A)dS_d$  is encoded in the following structure equations

$$d\mathbf{w}_{ab} + \mathbf{w}_a{}^c \wedge \mathbf{w}_{cb} = \Lambda \mathbf{e}_a \wedge \mathbf{e}_b, \quad (2.5)$$

$$D\mathbf{e}_a = d\mathbf{e}_a + \mathbf{w}_a{}^b \wedge \mathbf{e}_b = 0. \quad (2.6)$$

The corresponding curvature tensor  $R_{ab,cd} = \Lambda(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad})$  is of  $dS_d$  for  $\Lambda > 0$ ,  $AdS_d$  if  $\Lambda < 0$  or Minkowski for  $\Lambda = 0$ . Equations (2.5) and (2.6) have manifest local gauge symmetry

$$\delta\mathbf{w}_{ab} = D\xi_{ab} + \Lambda(\xi_a\mathbf{e}_b - \xi_b\mathbf{e}_a), \quad \delta\mathbf{e}_a = D\xi_a - \xi_{ab}\mathbf{e}^b, \quad (2.7)$$

where  $\xi_{ab} = -\xi_{ba}$  and  $\xi_a$  are arbitrary 0-forms. Any particular solution of (2.5), (2.6) breaks down its local symmetry. The leftover global symmetry is determined by  $\delta\mathbf{w}_{ab} = \delta\mathbf{e}_a = 0$ , equivalently

$$D\xi_a = \xi_{ab}\mathbf{e}^b, \quad (2.8)$$

$$D\xi_{ab} = -\Lambda(\xi_a\mathbf{e}_b - \xi_b\mathbf{e}_a). \quad (2.9)$$

The first equation (2.8) says that  $D_a\xi_b$  does not contain its symmetric part. In other words  $\xi_a$  is a Killing vector

$$D_a\xi_b + D_b\xi_a = 0. \quad (2.10)$$

Equation (2.9) is simply the  $(A)dS_d$  consistency condition arising from  $D^2\xi_a = \Lambda\mathbf{e}_a \wedge \mathbf{e}_b \xi^b$ .

Let us note that equation (2.9) alone (equation (2.8) is not imposed) being written in an arbitrary space-time imposes severe restrictions on its geometry. Those spaces which admit nontrivial (2.9) compatible with Bianchi identities possess hidden symmetries [11]. These symmetries arise from Yano-Killing tensors associated with  $\xi_{ab}$ . The field  $\xi_{ab}$  itself was called in [11] the principal conformal Yano-Killing tensor (CYK). It plays a crucial role in the variable separation problem for higher-dimensional black holes. Moreover, in [32]  $2d$ -dimensional Chen-Lü-Pope black holes [3] were shown to be the only on-shell solutions which admit CYK field. As the mass and NUT-parameter of a black hole are set to zero, the metric reduces to  $(A)dS_d$ .

Consider now the flat case with  $\Lambda = 0$ . It will be convenient to have the same form of equation (2.9) in this limit. However, as  $\Lambda \rightarrow 0$  one arrives at  $D\xi_{ab} = 0$ . A field redefinition  $(\xi_a, \xi_{ab}) \leftrightarrow (v_a, \Phi_{ab})$  implies

$$Dv_a = 0, \quad D\Phi_{ab} = v_a\mathbf{e}_b - v_b\mathbf{e}_a. \quad (2.11)$$

Equations in (2.11) are obviously consistent with  $D^2 = 0$  and have the same amount of fields as in (2.8), (2.9). This redefinition can be made explicit in the Cartesian reference frame, for example, with  $\mathbf{w}_{ab} = 0$ ,  $\mathbf{e}_a = dx_a$ :

$$v_a = \xi_a - \xi_{ab}x^b, \quad \Phi_{ab} = \xi_{ab} + (\xi_a - \xi_{ac}x^c)x_b - (\xi_b - \xi_{bc}x^c)x_a. \quad (2.12)$$

It is clear that (2.11) is still a covariant constancy condition for the Poincaré case. Note, that  $v^a$  is a Minkowski Killing vector. Not all Killing vectors satisfy  $Dv_a = 0$  in Minkowski space-time

though the rest - which do not - are encoded in  $\Phi_{ab}$ . Using these new fields one rewrites the global symmetry parameter equation for  $(A)dS_d$  or Minkowski space-time in a uniform manner that preserves the CYK equation<sup>5</sup>

$$Dv_a = -2\Lambda\Phi_{ab}\mathbf{e}^b, \quad (2.13)$$

$$D\Phi_{ab} = \frac{1}{2}(v_a\mathbf{e}_b - v_b\mathbf{e}_a). \quad (2.14)$$

For any finite  $\Lambda$  equations (2.13)-(2.14) are equivalent to (2.8)-(2.9) upon the notation change  $v_a = -2\Lambda\xi_a$ ,  $\Phi_{ab} = \xi_{ab}$  which gets degenerate if  $\Lambda = 0$ . Still, as was demonstrated, the system (2.13)-(2.14) admits well defined flat limit. These equations will be a starting point on the way of coordinate free formulation of  $5d$  black holes.

So far we have been considering generic  $d$ -dimensional case. Let us restrict ourselves further on  $d = 5$  space-time, *i.e.* vector indices range  $a, b = 0, \dots, 4$ . Most of the properties that we need in what follows are easily derived from the spinor form of (2.13)-(2.14).

### 3 Spinor analysis

The vector  $v_a$  has its antisymmetric traceless bispinor counterpart  $v_{\alpha\beta} = -v_{\beta\alpha}$  in the  $5d$  spinorial notation, while the antisymmetric tensor  $\Phi_{ab}$  is represented by the symmetric bispinor  $\Phi_{\alpha\beta} = \Phi_{\beta\alpha}$  (see appendix B). Analogously, fünfbein  $\mathbf{e}^a$  is given by the antisymmetric and traceless bispinor 1-form  $\mathbf{e}_{\alpha\beta} = -\mathbf{e}_{\beta\alpha}$ . As a result, equations (2.13)-(2.14) read

$$Dv_{\alpha\beta} = -\frac{\Lambda}{2}(\Phi_\alpha{}^\gamma\mathbf{e}_{\gamma\beta} - \Phi_\beta{}^\gamma\mathbf{e}_{\gamma\alpha}), \quad (3.1)$$

$$D\Phi_{\alpha\beta} = \frac{1}{2}(v_\alpha{}^\gamma\mathbf{e}_{\gamma\beta} + v_\beta{}^\gamma\mathbf{e}_{\gamma\alpha}). \quad (3.2)$$

This system has two independent Lorentz scalars which can choose to be

$$H = \sqrt{\det \Phi_{\alpha\beta}}, \quad Q = \frac{1}{4}\Phi_{\alpha\beta}\Phi^{\alpha\beta}. \quad (3.3)$$

For these, one finds

$$dH = -\frac{1}{2}H(\Phi^{-1})^{\alpha\beta}v_\alpha{}^\gamma\mathbf{e}_{\gamma\beta}, \quad dQ = \frac{1}{2}\Phi^{\alpha\beta}v_\alpha{}^\gamma\mathbf{e}_{\gamma\beta}. \quad (3.4)$$

The rest scalars are either expressed via these two through Fierz identities (D.5), (D.9)-(D.11) (see Appendix D) or get reduced to the following first integrals

$$P^2 = \frac{1}{4}v_{\alpha\beta}v^{\alpha\beta} + \Lambda Q = v^2 + \Lambda Q = \text{const}, \quad (3.5)$$

$$I_1 = -\frac{1}{2}\left(\frac{1}{4}\Phi_{\alpha\beta}\Phi_{\gamma\delta}v^{\alpha\gamma}v^{\beta\delta} + P^2 Q - \frac{\Lambda}{2}(Q^2 + H^2)\right) = \text{const}, \quad (3.6)$$

$$I_2 = \frac{i}{4}(\Phi^2)_{\alpha\beta}v^{\alpha\beta} = \text{const}, \quad (3.7)$$

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<sup>5</sup>Factors of two and one-half in (3.1)-(3.2) have been introduced for future convenience when  $d = 5$  and the equations rewritten in the spinor form.

It is straightforward to check using (3.1)-(3.2) that  $dP^2 = dI_{1,2} = 0$ . The number of first integrals is related to the rank of space-time algebra which is equal to three for either Poincaré,  $so(4,2)$  or  $so(5,1)$ . One can think of the fields  $v_{\alpha\beta}$  and  $\Phi_{\alpha\beta}$  as of the spinor representation of space-time algebra satisfying the zero-curvature condition (3.1)-(3.2). The first integrals (3.5)-(3.7) are hence the corresponding Casimir operators. Their vector form amounts to

$$P^2 = v_a v^a - \frac{\Lambda}{2} \Phi_{ab} \Phi^{ab}, \quad (3.8)$$

$$I_1 = 2(\Phi^2)_{ab} v^a v^b + \frac{1}{4} \Phi_{ab} \Phi^{ab} \cdot P^2 + \frac{\Lambda}{4} (Q^2 + H^2), \quad (3.9)$$

$$I_2 = \Phi_{ab} \Phi_{cd} v_e \varepsilon^{abcde}. \quad (3.10)$$

The system (3.1)-(3.2) possesses a number of remarkable properties. One of the most important is its relation to the solutions of massless field equations on  $(A)dS_5$ . The fields  $\Phi_{\alpha\beta}$  and  $v_{\alpha\beta}$  can be shown to generate the whole tower of integer spin solutions of  $(A)dS_5$  Fronsdal equations. Spin zero and spin one cases are easily derivable. Indeed, from (3.4) it is straightforward to obtain

$$\square \frac{1}{H} = 4 \frac{\Lambda}{H}. \quad (3.11)$$

Hence, the field  $\phi = \frac{1}{H}$  satisfy the Klein-Gordon equation. The mass-like term on the r.h.s. of (3.11) exactly corresponds to that of the massless scalar on  $(A)dS_5$ . Massless spin one field takes its origin from a source free Maxwell tensor, which can be constructed as follows. Consider

$$F_{\alpha\beta} = \frac{1}{H} \Phi_{\alpha\beta}^{-1}. \quad (3.12)$$

Using (3.1)-(3.2), one finds

$$dF_{\alpha\beta} = \frac{1}{2H} (F_{\alpha\beta} F_{\gamma\delta} + F_{\alpha\gamma} F_{\beta\delta} + F_{\alpha\delta} F_{\beta\gamma}) v^{\gamma\lambda} \mathbf{e}_\lambda{}^\delta. \quad (3.13)$$

The tensor  $S_{\alpha\beta\gamma\delta} = F_{\alpha\beta} F_{\gamma\delta} + F_{\alpha\gamma} F_{\beta\delta} + F_{\alpha\delta} F_{\beta\gamma}$  which appears in the parenthesis of (3.13) being totally symmetric entails the identities

$$D_{[\alpha\beta} F_{\gamma]\delta} = 0, \quad D_{\alpha\gamma} F_{\beta}{}^\gamma - D_{\beta\gamma} F_{\alpha}{}^\gamma = 0 \quad (3.14)$$

both equivalent in vector notation to Maxwell equations for the tensor  $F^{ab} = \frac{1}{8} \Gamma_{\alpha\beta}^{ab} F^{\alpha\beta}$ :

$$\partial_{[a} F_{bc]} = 0, \quad D_b F^b{}_a = 0. \quad (3.15)$$

Therefore, locally

$$F_{ab} = \partial_a A_b - \partial_b A_a, \quad \square A_a - D_b D_a A^b = 0. \quad (3.16)$$

To get an appropriate spin one potential  $A^a$  and proceed to higher spins we note that these massless fields are uniformly described by Kerr-Schild vectors.

### 3.1 Kerr-Schild vectors and massless fields

In this section we generalize the construction of [14] to the five-dimensional case. The idea of [14] was to generate Kerr-Schild vectors out of the Killing vector  $v^a$  by projecting it onto

appropriate light-like directions. These directions were associated with eigen spinors of the projectors made of the  $\Phi_{ab}$  field. Ranks of so-defined projectors were equal to one. The whole scheme was essentially four dimensional based on two-component spinors leaving but little possibility for higher-dimensional generalization. The desired generalization does exist after all as we are going to demonstrate. Its key element turns out to be Clifford algebra rather than two-component spinors.

Consider the following projectors to single out light-like vectors

$$\Pi_{\alpha\beta}^{\pm} = \frac{1}{2}(\epsilon_{\alpha\beta} \pm X_{\alpha\beta}), \quad (3.17)$$

where  $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$  is the charge conjugation matrix (see appendix B),  $X_{\alpha\beta} = X_{\beta\alpha}$  and

$$(X^2)_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta}. \quad (3.18)$$

Their straightforward properties are

$$\Pi_{\alpha}^{\pm\gamma}\Pi_{\gamma\beta}^{\pm} = \Pi_{\alpha\beta}^{\pm}, \quad \Pi_{\alpha}^{\pm\gamma}\Pi_{\gamma\beta}^{\mp} = 0, \quad \Pi_{\alpha\beta}^{+} = -\Pi_{\beta\alpha}^{-}. \quad (3.19)$$

From (3.19) it is obvious that the vectors

$$v_{\alpha\beta}^{+} = \Pi_{\alpha\gamma}^{+}\Pi_{\beta\delta}^{+}v^{\gamma\delta}, \quad v_{\alpha\beta}^{-} = \Pi_{\alpha\gamma}^{-}\Pi_{\beta\delta}^{-}v^{\gamma\delta}$$

are both light-like. The involutory matrix (3.18) is fixed exactly once being dependent on  $\Phi_{\alpha\beta}$  only. In that case, one readily finds

$$X_{\alpha\beta} = \frac{1}{2r}(\Phi_{\alpha\beta} + H\Phi_{\alpha\beta}^{-1}), \quad (3.20)$$

where

$$r^2 = \frac{1}{2}(H - Q). \quad (3.21)$$

The two Kerr-Schild vectors are given by

$$k_{\alpha\beta} = \frac{v_{\alpha\beta}^{+}}{v^{+}v^{-}}, \quad n_{\alpha\beta} = \frac{v_{\alpha\beta}^{-}}{v^{+}v^{-}}, \quad (3.22)$$

where

$$v^{+}v^{-} = \frac{1}{4}v_{\alpha\beta}^{+}v^{-\alpha\beta} = \frac{1}{4}\Pi_{\alpha\gamma}^{+}\Pi_{\beta\delta}^{+}v^{\gamma\delta}v^{\alpha\beta}. \quad (3.23)$$

The normalization is chosen such that  $k^av_a = n^av_a = 1$ . The vectors in (3.22) are null by definition. That these satisfy geodesity condition

$$k^a D_a k_b = n^a D_a n_b = 0 \quad (3.24)$$

is far from being obvious and deserves some more attention. Equation (3.24) can be proven using (3.1)-(3.2) and definitions (3.17), (3.20), (3.21) by direct differentiation along with the use of 5d Fierz identities. This calculation gives little perception as to what extent the proposed construction is general, though. In fact, property (3.24) is valid for null vectors based on similar projectors defined within generic Clifford algebra and does not require an explicit form of the involutory matrix  $X_{\alpha\beta}$  for its proof. Aiming possible generalization of our construction



to higher dimensions, we provide the proof of (3.24) which does not refer to formula (3.20) in appendix C.

Equations (3.1)-(3.2) are invariant under the discrete symmetry

$$\tau_c : v^a \rightarrow c \cdot v^a, \quad \Phi_{ab} \rightarrow c \cdot \Phi_{ab}, \quad (3.25)$$

where  $c$  is an arbitrary real constant. It interchanges Kerr-Schild vectors for  $c = -1$

$$k^a = \tau_{-1}(n^a). \quad (3.26)$$

Properly normalized difference between the two Kerr-Schild vectors is a total derivative. Indeed, it can be easily shown that

$$d\left(\frac{1}{H}(k_{\alpha\beta} - n_{\alpha\beta})\mathbf{e}^{\alpha\beta}\right) = 0. \quad (3.27)$$

Moreover, both vectors appear as potentials for the Maxwell tensor defined in (3.12)

$$4F_{\alpha\beta} = \partial_{\alpha\gamma}\left(\frac{1}{H}k^\gamma_\beta\right) + \partial_{\beta\gamma}\left(\frac{1}{H}k^\gamma_\alpha\right) = \partial_{\alpha\gamma}\left(\frac{1}{H}n^\gamma_\beta\right) + \partial_{\beta\gamma}\left(\frac{1}{H}n^\gamma_\alpha\right), \quad (3.28)$$

or, equivalently, in the vector form

$$4F_{ab} = \partial_b \frac{k_a}{H} - \partial_a \frac{k_b}{H} = \partial_b \frac{n_a}{H} - \partial_a \frac{n_b}{H}. \quad (3.29)$$

From Maxwell equations (3.15) we, therefore, obtain

$$\square\left(\frac{k_a}{H}\right) - D_b D_a \left(\frac{k^b}{H}\right) = 0. \quad (3.30)$$

Equations (3.11) and (3.30) correspond to massless scalar and spin  $s = 1$  equations on  $(A)dS_5$ , respectively. A natural candidate for the massless spin  $s = 2$  field is  $h_{ab} = \frac{1}{H}k_a k_b$ . It is straightforward if somewhat involved calculation based on (D.6)-(D.11) that confirms this guess yielding linearized Einstein equations

$$\square h_{mn} - D_p D_m h^p_n - D_p D_n h^p_m = -8\Lambda h_{mn}. \quad (3.31)$$

A sequence of Kerr-Schild massless fields naturally goes on as is clear from (3.11), (3.30), (3.31) and generates the spin- $s$  Fronsdal field

$$\phi_{a_1 \dots a_s} = \frac{1}{H} k_{a_1} \dots k_{a_s} \quad (3.32)$$

satisfying the  $(A)dS_5$  Fronsdal equations

$$\square \phi_{a_1 \dots a_s} - s D_b D_{(a_1} \phi_{a_2 \dots a_s)}^b = -2\Lambda(s-1)(s+2)\phi_{a_1 \dots a_s} \quad (3.33)$$

in agreement<sup>6</sup> for  $d = 5$  with [33]. To prove (3.33) it is sufficient to prove it for  $s = 0, 1, 2$ . The generic  $s$  case comes out as a straightforward consequence.

A comment is now in order. That geodesic shear-free congruence generates solutions of massless equations is natural in four dimensions as a consequence of the famous Penrose twistor transform [16]. Indeed, the poles of the integrand of the latter exactly reproduce Kerr-Schild congruences. To the best of our knowledge no such explanation is known for higher dimensions.

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<sup>6</sup>To obtain literal agreement with the result of Metsaev [33] for the case of the equations for totally symmetric Fronsdal fields in  $AdS$ , one needs to apply the Ricci identity  $[D_a, D_b]T^c = \Lambda(\delta_a^c T_b - \delta_b^c T_a)$  to change the order of derivatives in the second term on the l.h.s of (3.33).

## 4 Black holes

In accordance with the considerations of Section 2.2, the constructed null vectors (3.22) have all necessary properties to solve  $d = 5$  Einstein equations  $R_{mn} = 4\Lambda g_{mn}$  in the form (2.1). Any vector of (3.22) can be chosen as a Kerr-Schild vector in (2.1). The resulted metrics are equivalent as will be demonstrated. Let us choose  $k_{\alpha\beta}$  for definiteness. So

$$g_{mn} = \bar{g}_{mn} + \frac{2M}{H} k_m k_n, \quad (4.1)$$

where  $\bar{g}_{mn}$  is the  $(A)dS_5$  metric,  $M$  is an arbitrary constant,  $H$  is given by (3.3) and  $k_m$  is defined in (3.22), solves Einstein equations with the cosmological constant in coordinate independent fashion. Weyl tensor is calculated to be

$$C_{\alpha\beta\gamma\delta} = -\frac{32M}{H} (F_{\alpha\beta} F_{\gamma\delta} + F_{\alpha\gamma} F_{\beta\delta} + F_{\alpha\delta} F_{\beta\gamma}), \quad (4.2)$$

its type is 22 according to the De Smet classification [29].

Casimir invariants (3.5)-(3.7) are diffeomorphism invariant characterizing a set of *inequivalent* metrics for different values of constants  $P^2, I_1, I_2$ . Using the scale ambiguity (3.25) it is possible, however, to set *e.g.*,  $P^2$  discrete equal to either  $P^2 = -1, 0, 1$ . An arbitrary constant  $M$  thus restores back scale ambiguity of fixed  $P^2$ , unless  $P^2 = 0$  in which case  $M$  is no longer relevant and can be taken  $M = 1$ , for example. Eventually, the metrics (4.1) are fully characterized by one discrete parameter  $P^2$  and three continuous  $M, I_1, I_2$ . Let us demonstrate now that (4.1) is a five-dimensional analogue of  $d = 4$  Carter-Plebanski metric [4, 5] without electro-magnetic charges (note, that there are no NUT-charges in five dimensions). It contains Myers-Perry black hole rotating about two independent planes for  $P^2 = -1$ . The black hole mass is given by  $M$  while its angular momenta are encoded in  $I_1, I_2$ . The cases with  $P^2 = 1$  and  $P^2 = 0$  are novel in  $d = 5$  and can be interpreted as “tachyonic” and “light-like” black holes, respectively.

### 4.1 Explicit realization

To simplify the calculations, let us set  $\Lambda = 0$  to focus on black holes in Minkowski space-time. Let us now enlist all of the metrics explicitly in the Cartesian reference frame  $x^a = (t, x, y, z, u)$ . To do so, we write down the general solution of the equations (3.1)-(3.2)

$$v_{\alpha\beta} = v_{\alpha\beta}^0 = \text{const}, \quad \Phi_{\alpha\beta} = \frac{1}{2} (v_{\alpha}^{\gamma} x_{\gamma\beta} + v_{\beta}^{\gamma} x_{\gamma\alpha}) + \Phi_{\alpha\beta}^0, \quad \Phi_{\alpha\beta}^0 = \text{const}, \quad (4.3)$$

where  $x_{\alpha\beta} = x^a \gamma_{a\alpha\beta}$ . Using a convenient parametrization, all inequivalent solutions can be summarized in the following table

Type	Killing vector $v_{\alpha\beta}$	Lorentz generator $\Phi_{\alpha\beta}^0$	$P^2$	$I_1$	$I_2$
Kerr	$\frac{\partial}{\partial t}$	$a\Gamma_{\alpha\beta}^{xy} + b\Gamma_{\alpha\beta}^{zu}$	-1	$b^2 + a^2$	$2ab$
Light-like Kerr	$\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$	$a\Gamma_{\alpha\beta}^{xy} + b\Gamma_{\alpha\beta}^{zu}$	0	$a^2$	$2ab$
Tachyonic Kerr	$\frac{\partial}{\partial x}$	$a\Gamma_{\alpha\beta}^{ty} + b\Gamma_{\alpha\beta}^{zu}$	+1	$a^2 - b^2$	$2ab$

**Table 1.** Classification of Kerr-Schild solutions on 5d Minkowski space by its Poincaré invariants

## Kerr metric, $P^2 = -1$

To reproduce a Kerr black hole we choose  $v^a = (1, 0, 0, 0, 0)$  or equivalently  $v_{\alpha\beta} = \gamma_{0\alpha\beta}$  and

$$\Phi_{\alpha\beta}^0 = a\Gamma_{\alpha\beta}^{xy} + b\Gamma_{\alpha\beta}^{zu}, \quad (4.4)$$

where  $x^a = (t, x, y, z, u)$ . A straightforward calculation gives

$$\det \Phi_{\alpha\beta} = H^2, \quad (4.5)$$

where

$$H = \frac{1}{r^2}(r^2 + a^2)(r^2 + b^2) \left( 1 - \frac{a^2(x^2 + y^2)}{(r^2 + a^2)^2} - \frac{b^2(z^2 + u^2)}{(r^2 + b^2)^2} \right) \quad (4.6)$$

and  $r$  defined in (3.21) reads

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{u^2 + z^2}{r^2 + b^2} = 1. \quad (4.7)$$

Casimir invariants are

$$P^2 = -1, \quad I_1 = a^2 + b^2, \quad I_2 = 2ab. \quad (4.8)$$

Thus,

$$a = \frac{1}{2} \left( \sqrt{I_1 + I_2} + \sqrt{I_1 - I_2} \right), \quad b = \frac{1}{2} \left( \sqrt{I_1 + I_2} - \sqrt{I_1 - I_2} \right), \quad (4.9)$$

For the Kerr-Schild vector  $k_a$  we also find

$$k_0 = 1, \quad k_1 = -\frac{xr + ay}{r^2 + a^2}, \quad k_2 = -\frac{yr - ax}{r^2 + a^2}, \quad (4.10)$$

$$k_3 = -\frac{zr + bu}{r^2 + b^2}, \quad k_4 = -\frac{ur - bz}{r^2 + b^2}. \quad (4.11)$$

This case corresponds to the familiar 5d Myers-Perry metric [1]. The parameters  $a$  and  $b$  are the angular momenta per unit mass of a rotating black hole about the  $xy$ -plane and the  $zu$ -plane. Coordinates of the another Kerr-Schild vector  $n_a$  are reproduced upon sign flip  $(a, b, x, y, z, u \rightarrow -a, -b, -x, -y, -z, -u)$  in (4.10).

## Light-like Kerr, $P^2 = 0$

Another solution with  $P^2 = 0$  which can be called “light-like” Kerr has its 4d analog described by the Carter-Plebanski metric [4, 5] with the discrete Carter parameter  $\epsilon = 0$ . The value  $P^2 = 0$  can be reached by the vector field

$$v^a = (1, 1, 0, 0, 0). \quad (4.12)$$

Taking

$$\Phi_{\alpha\beta}^0 = a\Gamma_{\alpha\beta}^{xy} + b\Gamma_{\alpha\beta}^{zu}, \quad (4.13)$$

we obtain

$$P^2 = 0, \quad I_1 = a^2, \quad I_2 = 2ab. \quad (4.14)$$

Let us note that unlike Kerr, in this case  $I_1 = 0$  implies  $I_2 = 0$ . The equations (3.21) now reads

$$a^2 \frac{u^2 + z^2}{r^2 + b^2} + (t - x)^2 - (y + a)^2 = r^2 - y^2. \quad (4.15)$$

The function  $H$  and the Kerr-Schild vector  $k_a$  are

$$H = (r^2 + b^2) \left( 1 + a^2 \frac{u^2 + z^2}{(r^2 + b^2)^2} \right), \quad (4.16)$$

$$k_0 = 1 + \frac{r^2 + ay - r(t - x)}{a^2}, \quad k_1 = \frac{-r^2 - ay + r(t - x)}{a^2}, \quad (4.17)$$

$$k_2 = \frac{r + x - t}{a}, \quad k_3 = -\frac{bu + zr}{r^2 + b^2}, \quad k_4 = \frac{bz - ur}{b^2 + r^2}. \quad (4.18)$$

The physical interpretation of this solution is not straightforward. Its four dimensional analog suffers from pathologies in global properties [34], so we do not expect that the  $5d$  counterpart would be any better. The notion “light-like” is justified, however, by the asymptotic behavior of the metric. Unlike the Kerr case where the gravitational field at large distances is described by a point source on a time-like surface, the present case asymptotically corresponds to a light-like source.

## Tachyonic Kerr, $P^2 = 1$

This case is naturally described by

$$v^a = (0, 1, 0, 0, 0), \quad \Phi_{\alpha\beta}^0 = a\Gamma_{\alpha\beta}^{ty} + b\Gamma_{\alpha\beta}^{zu}. \quad (4.19)$$

The corresponding Poincaré invariants are

$$P^2 = 1, \quad I_1 = a^2 - b^2, \quad I_2 = 2ab. \quad (4.20)$$

The scalars  $r$ ,  $H$  and Kerr-Schild vector  $k_a$  take the following form

$$\frac{u^2 + z^2}{r^2 + b^2} + \frac{y^2 - t^2}{r^2 - a^2} = -1, \quad (4.21)$$

$$H = \frac{1}{r^2} (r^2 - a^2)(r^2 + b^2) \left( 1 - \frac{a^2(y^2 - t^2)}{(r^2 - a^2)^2} + \frac{b^2(u^2 + z^2)}{(r^2 + b^2)^2} \right), \quad (4.22)$$

$$k_0 = \frac{-ay + rt}{r^2 - a^2}, \quad k_1 = 1, \quad k_2 = \frac{at - yr}{r^2 - a^2}, \quad (4.23)$$

$$k_3 = -\frac{bu + zr}{r^2 + b^2}, \quad k_4 = \frac{bz - ur}{r^2 + b^2}. \quad (4.24)$$

This solution has a  $4d$  analog represented by the non-charged Carter-Plebanski metric with  $\epsilon = -1$ . Its physical relevance is not clear and it has some pathologies in the global structure [34]. Asymptotically metric (4.1) describes the gravitational field of a tachyonic source.

As it was already mentioned, the parameter  $P^2$  is analogous to  $d = 4$  Carter-Plebanski parameter  $\epsilon$ . For  $\Lambda \neq 0$  the proper normalization for Casimir invariants will be given in the next section.

## 4.2 Unfolded form of the black hole

In section 4, we have shown how black holes are generated by a single  $(A)dS_5$  global symmetry parameter in arbitrary coordinates. Apart from its coordinate independence the advantage of the proposed approach is that the fields which constitute black hole's metric satisfy the background  $(A)dS_5$  equation. The latter is just the zero curvature condition of  $so(4, 2)$  or  $so(5, 1)$  algebra thus being pure gauge. Black holes appear as the Kerr-Schild transform of the background metric (4.1). It is possible, however, to deform (3.1)-(3.2) such that its consistency condition  $\mathcal{D}^2 \sim R$ ,  $\mathcal{D}R = 0$  would imply black hole's curvature rather than  $(A)dS_5$ . Indeed, consider the following one-parametric deformation of (3.1)-(3.2)

$$\mathcal{D}v_{\alpha\beta} = -\frac{\Lambda}{2}(\Phi_\alpha^\gamma \mathbf{e}_{\gamma\beta} - \Phi_\beta^\gamma \mathbf{e}_{\gamma\alpha}) + \frac{\mu}{H}(\Phi_\alpha^{-1\gamma} \mathbf{e}_{\gamma\beta} - \Phi_\beta^{-1\gamma} \mathbf{e}_{\gamma\alpha}), \quad (4.25)$$

$$\mathcal{D}\Phi_{\alpha\beta} = \frac{1}{2}(v_\alpha^\gamma \mathbf{e}_{\gamma\beta} + v_\beta^\gamma \mathbf{e}_{\gamma\alpha}), \quad (4.26)$$

where  $\mu$  is an arbitrary real constant and  $H = \sqrt{\det \Phi_{\alpha\beta}}$ . This system is consistent, provided the Weyl tensor in (B.15) is given by

$$C_{\alpha\beta\gamma\delta} = -\frac{32\mu}{H^3}(\Phi_{\alpha\beta}^{-1}\Phi_{\gamma\delta}^{-1} + \Phi_{\alpha\gamma}^{-1}\Phi_{\beta\delta}^{-1} + \Phi_{\alpha\delta}^{-1}\Phi_{\beta\gamma}^{-1}). \quad (4.27)$$

It is straightforward to check that the Bianchi identity  $\mathcal{D}C_{\alpha\beta\gamma\delta} \wedge \mathbf{e}^{\gamma\lambda} \wedge \mathbf{e}_\lambda^\delta = 0$  holds. Equations (4.25)-(4.26) have clearly the unfolded form<sup>7</sup> as being consistent within given set of fields. Note, the equation (4.26) remains unchanged as compared to (3.2).

Analogously to [14], the deformed system (4.25)-(4.26) shares many properties with the nondeformed one (3.1)-(3.2). Particularly, as follows from (4.25),  $v_{\alpha\beta}$  is still a Killing vector. The field  $\Phi_{\alpha\beta}$  remains a principal CYK as well. It has the same number of first integrals. These are given by

$$I_0 = v^2 + \Lambda Q - \frac{2\mu}{H}, \quad (4.28)$$

$$I_1 = -\frac{1}{2}\left(\frac{1}{4}\Phi_{\alpha\beta}\Phi_{\gamma\delta}v^{\alpha\gamma}v^{\beta\delta} + I_0 Q - \frac{\Lambda}{2}(Q^2 + H^2)\right) + \mu, \quad (4.29)$$

$$I_2 = \frac{i}{4}(\Phi^2)_{\alpha\beta}v^{\alpha\beta}, \quad (4.30)$$

Note that the dimension of the mass parameter  $\mu$  in  $d = 5$  is  $[\mu] = 2$  and so is the dimension of  $I_{1,2}$ . Hence,  $\mu$  can be added up to arbitrary factors to  $I_1$  and  $I_2$ . A choice of that particular factor in  $I_1$  will become clear later on.

A space-time described by the equations (4.25)-(4.26) can be identified as follows. According to [11], Einstein spaces that admit the principal CYK tensor (4.26) should be of Chen-Lü-Pope type black holes [3]. In our five-dimensional case these were classified in the Table 1 for  $\Lambda = 0$ . The type of solution is defined by the sign of  $I_0$  invariant – Myers-Perry for  $I_0 < 0$ , “light-like” for  $I_0 = 0$  and “tachyonic” for  $I_0 > 0$ . Indeed, by appropriate global rescaling  $v \rightarrow cv$ ,  $\Phi \rightarrow c\Phi$ ,

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<sup>7</sup>Recall, the dynamical system is said to have the unfolded form if it is formulated in terms of first order differential equations for differential  $p$ -forms  $W^A$ ,  $p \geq 0$  and generalized curvatures  $R^A = dW^A + F^A(W)$ , provided the functions  $F^A$  are subject to the generalized Jacobi identities  $F^B \wedge \frac{\delta F^A}{\delta W^B} = 0$  that guarantee the generalized Bianchi identities  $dR^A = R^B \wedge \frac{\delta F^A}{\delta W^B}$  (see *e.g.*, [35]).

$\mu \rightarrow c^4 \mu$   $I_0$  can always be set to either  $-1, 0, 1$ . Hence, the sign of  $I_0$  distinguishes between inequivalent solutions.

Let us enlist some more properties of (4.25)-(4.26). It admits at least three Killing vectors which arise due to the existence of the CYK field  $\Phi_{\alpha\beta}$ . The first one,  $v_{\alpha\beta}$ , is manifest. To identify the other two, one constructs the Killing tensor  $K_{ab} = K_{ba}$  which produces the latter as  $\xi_a^{(1)} = K_{ab}v^b$  and  $\xi_a^{(2)} = K_{ab}\xi^{(1)b}$  (see [11] for more details). In the spinor notation the Killing tensor is represented by the following multispinor

$$K_{\alpha\beta,\gamma\delta} = K_{\gamma\delta,\alpha\beta}, \quad K_{\alpha\beta,\gamma\delta} = -K_{\beta\alpha,\gamma\delta} = -K_{\alpha\beta,\delta\gamma}, \quad K_{\alpha}{}^{\alpha},{}_{\gamma\delta} = K_{\alpha\beta,\gamma}{}^{\gamma} = 0.$$

In terms of CYK field  $\Phi_{\alpha\beta}$ , it has the following explicit form:

$$K_{\alpha\beta,\gamma\delta} = \Phi_{\alpha\gamma}\Phi_{\beta\delta} - \Phi_{\beta\gamma}\Phi_{\alpha\delta} + \frac{1}{2}(\Phi_{\gamma\delta}^2\epsilon_{\alpha\beta} + \Phi_{\alpha\beta}^2\epsilon_{\gamma\delta}) + \frac{1}{8}\Phi_{\mu\nu}\Phi^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}. \quad (4.31)$$

The corresponding Killing vectors then read

$$\xi_{\alpha\beta}^{(1)} = 2\Phi_{\alpha\gamma}\Phi_{\beta\delta}v^{\gamma\delta} + \frac{1}{2}\Phi_{\gamma\delta}^2v^{\gamma\delta}\epsilon_{\alpha\beta}, \quad \xi_{\alpha\beta}^{(2)} = 2\Phi_{\alpha\gamma}\Phi_{\beta\delta}\xi^{(1)\gamma\delta} + \frac{1}{2}\Phi_{\gamma\delta}^2\xi^{(1)\gamma\delta}\epsilon_{\alpha\beta}. \quad (4.32)$$

Recall now, that the  $(A)dS_5$  equations (3.1)-(3.2) admit the source-free Maxwell tensor (3.12). Introduction of the deformation parameter  $\mu$  leaves it unaffected, *i.e.*, equations (3.12) and (3.13) still hold. Same is the story with Kerr-Schild vectors. The whole projector construction and its properties (3.17)-(3.24) extend to black hole unfolded system upon the change  $D \rightarrow \mathcal{D}$  in (3.24). Pretty much as in the nondeformed case, Kerr-Schild vectors defined in (3.22) generate Maxwell potentials (3.28) resulting in the following Maxwell equations

$$\square\left(\frac{k_a}{H}\right) - \mathcal{D}_b\mathcal{D}_a\left(\frac{k^b}{H}\right) = 0. \quad (4.33)$$

One can equally well replace  $k^a$  with  $n^a$  in (4.33). Interestingly enough, even linearized Einstein equations (3.31) hold upon covariantization  $D \rightarrow \mathcal{D}$ . This can not be said about Fronsdal equations (3.33) for  $s > 2$  which naturally break down for  $\mu \neq 0$ . Technically, (3.33) would have been held had equation (3.11) been satisfied. This is not the case; instead for  $\mu \neq 0$  one finds

$$\square\frac{1}{H} = 4\frac{\Lambda}{H} + 8\mu\frac{Q}{H^4}, \quad (4.34)$$

where  $Q$  is defined in (3.3).

From algebraic standpoint, both the  $(A)dS_5$  and black hole systems are to much extent equivalent and coincide at  $\mu = 0$ . The equivalence becomes manifest upon field redefinition that maps one system to another. An instructive way to obtain this map is to perform an integrating flow  $\frac{\partial}{\partial\mu}$  from initial surface  $\mu = 0$  to some fixed finite value. The flow in general is governed by first order ordinary differential equations of the form

$$\frac{\partial}{\partial\mu}v_{\alpha\beta} = f_{\alpha\beta}^{(1)}(v, \Phi, \mu), \quad \frac{\partial}{\partial\mu}\Phi_{\alpha\beta} = f_{\alpha\beta}^{(2)}(v, \Phi, \mu), \quad \frac{\partial}{\partial\mu}\mathbf{e}_{\alpha\beta} = f_{\alpha\beta}^{(3)}(\mathbf{e}, v, \Phi, \mu), \quad (4.35)$$

where the functions  $f^{(i)}$  to be fixed by the integrability requirement

$$[d, \frac{\partial}{\partial\mu}] = 0, \quad (4.36)$$

where  $d$  – is the space-time differential. Solving the evolution equations (4.35) one finds the searched for map. This strategy has been accomplished for  $d = 4$  black holes in [14]. Using similar approach we propose the following integrating flow for  $d = 5$  black holes

$$\frac{\partial}{\partial \mu} \Phi_{\alpha\beta} = 0, \quad \frac{\partial}{\partial \mu} v_{\alpha\beta} = \frac{1}{H} k_{\alpha\beta}, \quad \frac{\partial}{\partial \mu} \mathbf{e}_{\alpha\beta} = \frac{1}{4H} k_{\alpha\beta} k_{\gamma\delta} \mathbf{e}^{\gamma\delta}. \quad (4.37)$$

Equations (4.37) are motivated by the Kerr-Schild shift and turn out to be consistent with (4.36). They can be easily integrated if one notices that the Kerr-Schild vector  $k^a$  is constant along the flow. Indeed, applying (4.37) to its definition (3.22) we obtain  $\frac{\partial}{\partial \mu} k_{\alpha\beta} = 0$ . Note, that this is not true for  $n^a$ , *i.e.*,  $\frac{\partial}{\partial \mu} n_{\alpha\beta} \neq 0$ . As a result, the solution of (4.37) reads

$$\Phi_{\alpha\beta} = \Phi_{\alpha\beta}^0, \quad v_{\alpha\beta} = v_{\alpha\beta}^0 + \frac{\mu}{H} k_{\alpha\beta}^0, \quad \mathbf{e}_{\alpha\beta} = \mathbf{e}_{\alpha\beta}^0 + \frac{\mu}{4H} k_{\alpha\beta}^0 k_{\gamma\delta}^0 \mathbf{e}^{0\gamma\delta}, \quad (4.38)$$

where the subscript 0 is assigned to  $\mu = 0$  nondeformed fields of (3.1)-(3.2). Recall that  $k_a = k_a^0$ . The first integrals (4.28)-(4.30) are invariant along the flow as well  $\frac{\partial}{\partial \mu} I_{0,1,2} = 0$ , thanks to the additional  $\mu$  added to (4.29):

$$I_{0,1,2} = I_{0,1,2}^0. \quad (4.39)$$

Black hole parameters such as angular momenta and the Carter-Plebański parameter are therefore encoded in Casimir invariants (3.5)-(3.7). The metric can be now easily calculated

$$ds^2 = \frac{1}{4} \mathbf{e}_{\alpha\beta} \cdot \mathbf{e}^{\alpha\beta} = ds_0^2 + \frac{2\mu}{H} k_m k_n dx^m dx^n. \quad (4.40)$$

For the physically important solution obtained by Hawking, Hunter and Taylor-Robinson [28] that is for  $d = 5$  Myers-Perry black hole in the presence of a non-zero cosmological constant, the identification of the first integrals (4.28)-(4.30) along with the mass parameter  $\mu$  is found to be as follows

$$I_0 = -1 + \Lambda(a^2 + b^2), \quad (4.41)$$

$$I_1 = a^2 + b^2 - \Lambda a^2 b^2, \quad (4.42)$$

$$I_2 = 2ab, \quad (4.43)$$

$$\mu = M. \quad (4.44)$$

In deriving this result, we have compared the horizon equation  $\Delta(r) = 0$  of [28] to the horizon condition in our approach. To find the latter, we note that the scalar product of two geodesic light-like congruences is degenerate on the horizon. Hence, it is sufficient to analyze scalar product of two Kerr-Schild vectors (3.22) defined for (4.25)-(4.26) which is given by

$$k_a n^a = \frac{1}{v^+ v^-} = \frac{2H}{\Delta_r}, \quad (4.45)$$

where

$$\Delta_r = I_0 r^2 + 2\mu + \Lambda r^4 - I_1 - \frac{I_2^2}{4r^2}. \quad (4.46)$$

Comparing  $\Delta_r$  with that of [28], one reproduces (4.41)-(4.44).

## 5 Conclusion

Five-dimensional Myers-Perry type black hole have been reconsidered in a coordinate independent way based on unfolded description of dynamical systems. In addition to the cosmological constant that was accounted for by Hawking, Hunter, and Taylor-Robinson [28], we also include the analogue of the  $4d$  discrete Carter-Plebanski parameter to the solution. So-defined black holes acquire a beautiful algebraic classification. The whole class is generated by a background space-time (either  $(A)dS_5$  or Minkowski) single global symmetry parameter. Three Casimir invariants  $P^2, I_1, I_2$  associated with that parameter distinguish between inequivalent black holes and produce black hole's "hair" – the mass and angular momenta. The classification resembles that of relativistic fields with spins and masses originated from Casimir operators of  $AdS/Poincaré$  algebra. Ordinary black holes naturally arise this way for  $P^2 < 0$  along with the light-like ( $P^2 = 0$ ) and tachyonic ( $P^2 > 0$ ). The value of  $P^2$  can be associated with the  $5d$  analogue of the Carter-Plebanski parameter. The other two invariants  $I_1, I_2$  determine angular momenta. In this respect, it should be noted that some indication that black holes can be treated within representation theory was given in [37] at the linearized level. That black holes are generated by a background global symmetry parameter was shown in [14] for  $d = 4$  black holes. The analysis of [14], however, leaves no indication if the proposed classification extends to higher dimensions or not. Now when it is shown that it actually does for  $d = 5$ , we believe it could be extended to arbitrary dimension with the use of Clifford algebra. Indeed, the proposed construction is based on Kerr-Schild ansatz which mysteriously works for Myers-Perry black holes, while the very Kerr-Schild vector arises from spinor projectors as demonstrated.

Parallel to the result of [14] where the unfolded form of the  $4d$  Carter-Plebanski black hole has been given as a deformation of  $AdS_4$  global symmetry condition, a similar one-parametric deformation takes place for  $d = 5$  rotating black holes without electro-magnetic charges. The deformed equations are related to the vacuum ones via the integrating flow describing evolution with respect to black hole mass. The integration of the flow equations with the initial data corresponding to  $(A)dS_5$  space allows us to express black hole fields that describe its metric in terms of their vacuum values. It should be stressed that in both four and five dimensions the crucial element that makes deformation possible is the existence of a principal conformal Killing-Yano tensor<sup>8</sup> for Myers-Perry type black holes. Moreover, it is this field attributed to all higher-dimensional Myers-Perry type black holes that makes variable separation for Klein-Gordon and Dirac equations in black hole background possible [11].

Apart from elucidating the structure of  $d = 5$  black holes, one of the goals of the present research was an application of higher spin gauge theory machinery to describe black holes as it hopefully allows us to generalize the latter to include higher-spin interactions. In this respect, the analogous  $4d$  result of [14] turned out to be very instructive and indeed allowed us to find a  $4d$  higher-spin black hole solution [17]. We hope it will be possible to generalize the  $5d$  black hole in higher-spin theory as well. Indeed, the developed approach based on a background global symmetry parameter provides Kerr-Schild solutions for all free massless fields rather than  $s = 2$  gravity only. The interaction between these massless fields can in principal be accounted by the nonlinear higher-spin equations [36]. Little chance to do it straightforwardly though, since the spinor form of these equations in  $d = 5$  is lacking as yet.

The other motivation to focus on  $d = 5$  was the fact that it is the minimal space-time

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<sup>8</sup>In [14] black hole deformation was carried out in terms of Killing and Maxwell fields. The latter is related to principal CYK.



dimension where black holes of nonspherical topology arise [38]. Unfortunately the results of the paper give no hint on whether black rings admit a similar description based on a single global symmetry parameter of Minkowski space-time. One thing is for certain, even if they really do, their unfolded equations would be completely different as compared to those considered in our work. From the algebraic point of view, the black ring essentially differs from the spherical black hole. Particularly, its Weyl tensor is not algebraically special and it does not admit the principal CYK field. Still one may ask oneself if there is a Ricci flat consistent deformation of the Minkowski global symmetry condition within the same set of fields. 5d spinor language seems well adopted to tackle this problem.

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## Appendix

### A Cartan formalism

We use the following definition for the Riemann tensor

$$[D_m, D_n]T^k = R^k_{p,mn}T^p \quad (\text{A.1})$$

or in terms of Christoffel symbols

$$R^k_{l,mn} = \partial_m \Gamma^k_{ln} - \partial_n \Gamma^k_{lm} + \Gamma^p_{ln} \Gamma^k_{pm} - \Gamma^p_{lm} \Gamma^k_{pn} \quad (\text{A.2})$$

which are given by

$$\Gamma^l_{mn} = \frac{1}{2}g^{lp}(\partial_n g_{mp} + \partial_m g_{np} - \partial_p g_{mn}), \quad D_n T^m = \partial_n T^m + \Gamma^m_{np} T^p. \quad (\text{A.3})$$

The vacuum Einstein equations in five dimensions ( $m, n = 0, \dots, 4$ ) read

$$R_{mn} = 4\Lambda g_{mn}. \quad (\text{A.4})$$

A convenient way to describe space-time geometry is to use the Cartan formalism. Since it is used in our analysis, we recall it somewhat in detail. To proceed, introduce the antisymmetric Lorentz connection one-form  $dx^m \mathbf{w}_{ab,m} = -dx^m \mathbf{w}_{ba,m}$  and fünfbein one-form  $dx^m \mathbf{e}_{ab,m}$ , where fiber indices range  $a, b = 0, \dots, 4$ . Flat indices are raised and lowered by mostly plus Minkowski metric  $\eta_{ab}$ . Cartan-Maurer equations have the form

$$\mathbf{R}_{ab} = d\mathbf{w}_{ab} + \mathbf{w}_a^c \wedge \mathbf{w}_{cb}, \quad (\text{A.5})$$

$$\mathbf{R}_a = \mathcal{D}\mathbf{e}_a = d\mathbf{e}_a + \mathbf{w}_a^b \wedge \mathbf{e}_b = 0, \quad (\text{A.6})$$

where  $\mathbf{R}_{ab} = \mathbf{R}_{ab,mn} dx^m \wedge dx^n$  is the curvature two-form to be identified with the Riemann tensor as follows

$$\mathbf{R}_{ab} = \frac{1}{2} R_{mn,kl} \mathbf{e}_a^m \mathbf{e}_b^n dx^k \wedge dx^l, \quad (\text{A.7})$$

provided the frame field  $\mathbf{e}_a$  defines the metric  $g_{mn} = \mathbf{e}_{a,m} \mathbf{e}_{b,n} \eta^{ab}$ . Equation (A.6) is the metric postulate that sets torsion two-form  $\mathbf{R}_a$  to zero. The integrability condition  $d^2 = 0$  for (A.5)-(A.6) amounts to Bianchi identities

$$\mathcal{D}^2 T_a = \mathbf{R}_{ab} T^b, \quad \mathcal{D} \mathbf{R}_{ab} \wedge \mathbf{e}^b = 0, \quad (\text{A.8})$$

where  $T^a$  is an arbitrary vector. For  $(A)dS_5$  space-time, for example, the curvature two-form is  $R_{ab} = \Lambda \mathbf{e}_a \wedge \mathbf{e}_b$ . Einstein equations (A.4) imply that curvature is equal to that of  $(A)dS_5$  up to a totally traceless tensor

$$\mathbf{R}_{ab} = \Lambda \mathbf{e}_a \wedge \mathbf{e}_b + \frac{1}{2} C_{ab,cd} \mathbf{e}^c \wedge \mathbf{e}^d, \quad (\text{A.9})$$

where  $C_{ab,cd}$  is the traceless Weyl tensor written in fiber components. The integrability condition for Einstein spaces is, therefore,

$$\mathcal{D}^2 T_a = \Lambda \mathbf{e}_a \wedge \mathbf{e}_b T^b + \frac{1}{2} C_{ab,cd} T^b \mathbf{e}^c \wedge \mathbf{e}^d, \quad \mathcal{D} C_{ab,cd} \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d = 0. \quad (\text{A.10})$$

## B Spinors in five dimensions

Consider Clifford algebra in five dimensions

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}. \quad (\text{B.1})$$

Its dimension is  $2^{[5/2]} = 4$  and hence  $5d$   $\gamma$ -matrices can be realized by four dimensional as follows

$$\gamma_a = (\gamma_{\hat{a}}, i\gamma_5), \quad (\text{B.2})$$

where  $\hat{a} = 0, \dots, 3$  and  $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ . A convenient choice for  $4d$   $\gamma$ -matrices is the Majorana representation

$$\gamma_0 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \quad (\text{B.3})$$

$$\gamma_4 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad (\text{B.4})$$

where  $\sigma^{1,2,3}$  are Pauli matrices. Restoring spinor indices, we use the  $\gamma_{a\alpha}{}^\beta$ -notation for  $\gamma$ -matrices,  $\alpha, \beta = 1, \dots, 4$ . The charge conjugation matrix  $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$  is antisymmetric and is used to raise and lower spinor indices according to the rule

$$\gamma_{a\alpha\beta} = \gamma_{a\alpha}{}^\delta \epsilon_{\delta\beta}, \quad \gamma_a^{\alpha\beta} = \gamma_{a\delta}{}^\beta \epsilon^{\alpha\delta}, \quad \epsilon_{\alpha\delta} \epsilon^{\beta\delta} = \delta_\alpha{}^\beta. \quad (\text{B.5})$$

$\gamma$ -matrices are antisymmetric with respect to charge conjugation, that is,  $\gamma_{a\alpha\beta} = -\gamma_{a\beta\alpha}$ . Irreducibility, in addition, implies tracelessness condition  $\gamma_{a,\alpha}{}^\alpha = 0$ . Thus, the number of spinorial components of the traceless antisymmetric matrix  $\gamma_{a\alpha\beta}$  is equal to  $\frac{4(4-1)}{2} - 1 = 5$  coinciding with the number of components of the  $5d$  vector. This makes it possible to assign a vector to any antisymmetric and traceless bispinor and vice versa

$$T_{\alpha\beta} = T^a \gamma_{a\alpha\beta}, \quad T^a = \frac{1}{4} \gamma_{\alpha\beta}^a T^{\alpha\beta}. \quad (\text{B.6})$$

Consider now spinor representation for Lorentz generators

$$(\Gamma_{ab})_{\alpha\beta} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)_{\alpha\beta}. \quad (\text{B.7})$$

These are symmetric  $\Gamma_{\alpha\beta}^{ab} = \Gamma_{\beta\alpha}^{ab}$  and for fixed  $a$  and  $b$  have 10 spinor components as well as for fixed  $\alpha$  and  $\beta$  there are 10 vector components. Using now that the following antisymmetrized tensor product  $\gamma_{\alpha\beta}^a\gamma_{\gamma\delta}^b - \gamma_{\alpha\beta}^b\gamma_{\gamma\delta}^a$  has again the same number of components for either fixed vector or spinor indices one easily arrives at the identity

$$\gamma_{\alpha\beta}^a\gamma_{\gamma\delta}^b - \gamma_{\alpha\beta}^b\gamma_{\gamma\delta}^a = \epsilon_{\alpha\gamma}\Gamma_{\beta\delta}^{ab} - \epsilon_{\beta\gamma}\Gamma_{\alpha\delta}^{ab} - \epsilon_{\alpha\delta}\Gamma_{\beta\gamma}^{ab} + \epsilon_{\beta\delta}\Gamma_{\alpha\gamma}^{ab}, \quad (\text{B.8})$$

which allows us to establish one to one correspondence between antisymmetric tensor  $T_{ab} = -T_{ba}$  and its symmetric bispinor counterpart  $T_{\alpha\beta} = T_{\beta\alpha}$ :

$$T_{\alpha\beta} = \Gamma_{\alpha\beta}^{ab}T_{ab}, \quad T^{ab} = \frac{1}{8}\Gamma_{\alpha\beta}^{ab}T^{\alpha\beta}. \quad (\text{B.9})$$

The advantage of  $5d$  spinors becomes especially vivid when the Weyl tensor  $C_{ab,cd}$  is concerned. Being a window-like traceless diagram in tensorial terms, it has 35 independent components. Converting it with two  $\Gamma$ 's one obtains

$$C_{ab,cd}\Gamma_{\alpha\beta}^{ab}\Gamma_{\gamma\delta}^{cd} = C_{(\alpha\beta\gamma\delta)} + \dots, \quad (\text{B.10})$$

where we have extracted a totally symmetric multispinor on the r.h.s. of (B.10) such that ... contain the other irreducible parts of the spinor decomposition. However, the number of components of totally symmetric  $C_{\alpha\beta\gamma\delta}$  is equal to  $\frac{4 \cdot 5 \cdot 6 \cdot 7}{4!} = 35$ . As a result, the  $5d$  Weyl tensor is completely represented by the totally symmetric multispinor  $C_{\alpha\beta\gamma\delta}$

$$C_{\alpha\beta\gamma\delta} = C_{ab,cd}\Gamma_{\alpha\beta}^{ab}\Gamma_{\gamma\delta}^{cd}, \quad C_{ab,cd} = \frac{1}{64}C_{\alpha\beta\gamma\delta}\Gamma_{ab}^{\alpha\beta}\Gamma_{cd}^{\gamma\delta}. \quad (\text{B.11})$$

To rewrite Cartan equations (A.5)-(A.6) in the spinor form, we introduce the spinor symmetric Lorentz connection one-form  $\mathbf{w}_{\alpha\beta} = \Gamma_{\alpha\beta}^{ab}\mathbf{w}_{ab}$  and the antisymmetric traceless fünfbein one-form  $\mathbf{e}_{\alpha\beta} = \gamma_{\alpha\beta}^a\mathbf{e}_a$ . Equations (A.5)-(A.6) then read

$$\mathbf{R}_{\alpha\beta} = d\mathbf{w}_{\alpha\beta} + \frac{1}{4}\mathbf{w}_\alpha{}^\gamma \wedge \mathbf{w}_{\gamma\beta}, \quad (\text{B.12})$$

$$\mathcal{D}\mathbf{e}_{\alpha\beta} = d\mathbf{e}_{\alpha\beta} + \frac{1}{4}\mathbf{w}_\alpha{}^\gamma \wedge \mathbf{e}_{\gamma\beta} + \frac{1}{4}\mathbf{w}_\beta{}^\gamma \wedge \mathbf{e}_{\alpha\gamma} = 0, \quad (\text{B.13})$$

where  $\mathbf{R}_{\alpha\beta} = \mathbf{R}_{ab}\Gamma_{\alpha\beta}^{ab}$  and  $\mathcal{D}\xi_\alpha = d\xi_\alpha + \frac{1}{4}\mathbf{w}_\alpha{}^\beta\xi_\beta$ . The integrability condition (A.8) reduces to

$$\mathcal{D}^2\xi_\alpha = \frac{1}{4}\mathbf{R}_\alpha{}^\beta\xi_\beta, \quad \mathbf{R}_\alpha{}^\gamma \wedge \mathbf{e}_{\gamma\beta} - \mathbf{R}_\beta{}^\gamma \wedge \mathbf{e}_{\alpha\gamma} = 0, \quad (\text{B.14})$$

while (A.9) and (A.10) to, correspondingly,

$$\mathbf{R}_{\alpha\beta} = \Lambda\mathbf{E}_{\alpha\beta} + \frac{1}{16}C_{\alpha\beta\gamma\delta}\mathbf{E}^{\gamma\delta}, \quad (\text{B.15})$$

$$\mathcal{D}^2\xi_\alpha = \frac{1}{4}\Lambda\mathbf{E}_\alpha{}^\beta\xi_\beta - \frac{1}{64}C_{\alpha\beta\gamma\delta}\xi^\beta\mathbf{E}^{\gamma\delta}, \quad (\text{B.16})$$

where  $\mathbf{E}_{\alpha\beta} = \mathbf{E}_{\beta\alpha} = \mathbf{e}_\alpha^\gamma \wedge \mathbf{e}_{\gamma\beta}$ . The following property resulting from (B.8) has been used

$$\mathbf{e}_{\alpha\beta} \wedge \mathbf{e}_{\gamma\delta} = \frac{1}{2}(\epsilon_{\alpha\gamma}\mathbf{E}_{\beta\delta} - \epsilon_{\beta\gamma}\mathbf{E}_{\alpha\delta} - \epsilon_{\alpha\delta}\mathbf{E}_{\beta\gamma} + \epsilon_{\beta\delta}\mathbf{E}_{\alpha\gamma}). \quad (\text{B.17})$$

In establishing the spinor representation for Lorentz tensors one has to force the reality condition which has not been considered so far. Using that Hermitian conjugation for  $\gamma$ -matrices can be expressed as

$$\gamma_a^\dagger{}_\alpha{}^\beta = (\gamma_0\gamma_a\gamma_0)_\alpha{}^\beta, \quad (\Gamma_{ab}^\dagger)_\alpha{}^\beta = (\gamma_0\Gamma_{ab}\gamma_0)_\alpha{}^\beta \quad (\text{B.18})$$

we introduce the transformation

$$T_{\alpha\beta}^D = (\gamma_0 T \gamma_0)_{\alpha\beta}. \quad (\text{B.19})$$

Now,  $T_{\alpha\beta} = -T_{\beta\alpha}$  corresponds to the real vector  $T^a$  given it is traceless and

$$T_{\alpha\beta}^\dagger = T_{\alpha\beta}^D. \quad (\text{B.20})$$

Analogously,  $F_{\alpha\beta} = F_{\beta\alpha}$  is equivalent to the real antisymmetric tensor  $F_{ab} = -F_{ba}$  if

$$F_{\alpha\beta}^\dagger = F_{\alpha\beta}^D. \quad (\text{B.21})$$

Note, that a three-form  $B_{abc} = B_{[abc]}$  has a symmetric counterpart  $B_{\alpha\beta} = B_{\beta\alpha}$ ; however, the reality condition is different, namely  $B_{\alpha\beta}^\dagger = -B_{\alpha\beta}^D$ . This means, in particular, that the Hodge dualization  $*F_{abc} = \varepsilon_{abcde}F^{de}$  is reached by  $*F_{\alpha\beta} = iF_{\alpha\beta}$  in the spinor notation.

## C Proof of the geodesity condition

Let us prove (3.24) for the Kerr-Schild vector  $k^a$

$$k^a D_a k_b = 0. \quad (\text{C.1})$$

The proof for  $n^a$  is analogous. Before we start, one comment is in order. In what follows, we only need (3.2) of the two main equations (3.1)-(3.2). We also require  $v^a$  to be a Killing vector. This means that the geodesity condition will be valid for any covariant derivatives in (C.1) either  $D$  or  $\mathcal{D}$ .

The main idea for proving (C.1) is first to prove that the vector  $t^a = k^b D_b k^a$  is light-like, *i.e.*,

$$t^a t_a = 0. \quad (\text{C.2})$$

In Lorentz signature, two orthogonal light-like vectors are proportional. Since, by definition,  $t_a k^a = 0$  it implies

$$k^b D_b k_a \sim k_a, . \quad (\text{C.3})$$

The unknown factor can be found by converting (C.3) with  $v^a$ . Using that  $D_a v_b + D_b v_a = 0$  and  $v^a k_a = \text{const} = 1$  the factor is fixed to be zero.

To prove (C.2) we need an auxiliary lemma:

$$\Phi_{ab} k^b \sim k_a \quad (\text{C.4})$$

or in the spinor form

$$s_{\alpha\beta} \equiv \Phi_\alpha^\gamma k_{\gamma\beta} - \Phi_\beta^\gamma k_{\gamma\alpha} = A k_{\alpha\beta}, \quad (\text{C.5})$$

where  $A$  is some factor. Since  $s_{\alpha\beta} k^{\alpha\beta} = 0$ , it suffices to prove that  $s_{\alpha\beta} s^{\alpha\beta} = 0$ . From (3.22) it follows  $\Pi_\alpha^{-\gamma} k_{\gamma\beta} = 0$  or

$$X_\alpha^\gamma k_{\gamma\beta} = k_{\alpha\beta}, \quad (\text{C.6})$$

where  $X_{\alpha\beta}$  was defined in (3.20). Multiplying (C.5) by  $X_\delta^\alpha$  we have

$$X_\delta^\alpha s_{\alpha\beta} = X_\delta^\alpha \Phi_\alpha^\gamma k_{\gamma\beta} - \Phi_\beta^\gamma X_\delta^\alpha k_{\gamma\alpha}. \quad (\text{C.7})$$

Noting that  $X_\delta^\alpha \Phi_{\alpha\gamma} = -X_\gamma^\alpha \Phi_{\alpha\delta}$  because  $(X\Phi)_{\alpha\beta}$  contains only even powers of  $\Phi$  which are antisymmetric and making use of (C.6), equation (C.7) amounts to

$$X_\alpha^\gamma s_{\gamma\beta} = s_{\alpha\beta}. \quad (\text{C.8})$$

Squaring it, one immediately obtains  $s_{\alpha\beta} s^{\alpha\beta} = 0$  that proves (C.5).

Now we are in a position to prove the geodesity condition (C.1). This results from the following relations

$$\begin{aligned} \Phi_\alpha^\gamma t_{\gamma\beta} - \Phi_\beta^\gamma t_{\gamma\alpha} &= \frac{1}{4} (\Phi_\alpha^\gamma k^{\mu\nu} D_{\mu\nu} k_{\gamma\beta} - (\alpha \leftrightarrow \beta)) = \\ &= \frac{1}{4} \left( k^{\mu\nu} D_{\mu\nu} (\Phi_\alpha^\gamma k_{\gamma\beta}) - k^{\mu\nu} k_{\gamma\beta} D_{\mu\nu} \Phi_\alpha^\gamma - (\alpha \leftrightarrow \beta) \right) \stackrel{(\text{C.5}), (3.2)}{=} \\ &= \frac{1}{4} k^{\mu\nu} D_{\mu\nu} (A k_{\alpha\beta}) + k_\alpha^\gamma k_\beta^\delta v_{\gamma\delta} = A t_{\alpha\beta} + k_{\alpha\beta} k^a D_a A + k_\alpha^\gamma k_\beta^\delta v_{\gamma\delta} \end{aligned} \quad (\text{C.9})$$

The bispinor  $u_{\alpha\beta} = k_\alpha^\gamma k_\beta^\delta v_{\gamma\delta}$  is traceless, antisymmetric and yet satisfy the reality condition (B.20), thus, being a vector. It is orthogonal to  $k_{\alpha\beta}$  and is light-like; hence  $u_{\alpha\beta} \sim k_{\alpha\beta}$ . As a result (C.9) amounts to

$$\Phi_\alpha^\gamma t_{\gamma\beta} - \Phi_\beta^\gamma t_{\gamma\alpha} = A t_{\alpha\beta} + B k_{\alpha\beta}, \quad (\text{C.10})$$

where  $B$  is some irrelevant coefficient. From (C.10) it follows  $t_{\alpha\beta} t^{\alpha\beta} = 0$  thus concluding the proof of (C.1). Let us stress that the presented proof relies on the Lorentz signature. Therefore, the spinor reality condition plays an important role. We have omitted its explicit check here.

## D Useful identities

Many useful relations that have been used throughout the scope of the paper are mere consequences of Fierz identities. Namely, antisymmetrization over four spinorial indices is proportional to Levi-Cevita symbol  $\varepsilon_{\alpha\beta\gamma\delta}$  being expressed as the antisymmetrization of the product of two charge-conjugation matrices

$$\varepsilon_{\alpha\beta\gamma\delta} \sim (\epsilon_{\alpha\beta} \epsilon_{\gamma\delta} - \epsilon_{\gamma\beta} \epsilon_{\alpha\delta} - \epsilon_{\delta\beta} \epsilon_{\gamma\alpha}) \quad (\text{D.1})$$

Particularly, for a vector  $v_{\alpha\beta}$  one easily finds

$$v_{\alpha\beta} v_{\gamma\delta} - v_{\gamma\beta} v_{\alpha\delta} - v_{\delta\beta} v_{\gamma\alpha} = -v^2 (\epsilon_{\alpha\beta} \epsilon_{\gamma\delta} - \epsilon_{\gamma\beta} \epsilon_{\alpha\delta} - \epsilon_{\delta\beta} \epsilon_{\gamma\alpha}), \quad (\text{D.2})$$

$$v_\alpha^\beta v_{\beta\gamma} = v^2 \epsilon_{\alpha\gamma}, \quad v^2 = \frac{1}{4} v_{\alpha\beta} v^{\alpha\beta}. \quad (\text{D.3})$$

Another useful formula that can be obtained in this fashion relates  $\Phi_{\alpha\beta}^3$  to  $\Phi_{\alpha\beta}$  and  $\Phi_{\alpha\beta}^{-1}$

$$\Phi_{\alpha\beta}^3 + 2Q\Phi_{\alpha\beta} + H^2\Phi_{\alpha\beta}^{-1} = 0. \quad (\text{D.4})$$

It also allows us to express the scalar  $\Phi_{\alpha\beta}^{-1}\Phi^{-1\alpha\beta}$  via  $H$  and  $Q$  introduced in (3.3)

$$\frac{1}{4}\Phi_{\alpha\beta}^{-1}(\Phi^{-1})^{\alpha\beta} = \frac{Q}{H^2}. \quad (\text{D.5})$$

Here we put the other identities of the sort we have used:

$$\frac{1}{2}(\Phi_{\alpha}^{2\gamma}k_{\gamma\beta} - \Phi_{\beta}^{2\gamma}k_{\gamma\alpha}) = -Qk_{\alpha\beta}, \quad (\text{D.6})$$

$$k^a D_a \frac{1}{H} = \frac{2r}{H^2}, \quad D_a \left( \frac{rk^a}{H^3} \right) = -\frac{2Q}{H^4}, \quad (\text{D.7})$$

$$F_{ab}k^b = \frac{r}{2H^2}k_a, \quad d\left(\frac{k_{\alpha\beta}}{H}\mathbf{e}^{\alpha\beta}\right) = -F_{\alpha\beta}\mathbf{E}^{\alpha\beta}, \quad (\text{D.8})$$

$$Qv_{\alpha\beta} + \frac{1}{2}(\Phi_{\alpha}^{2\gamma}v_{\gamma\beta} - \Phi_{\beta}^{2\gamma}v_{\gamma\alpha}) = -iI_2\epsilon_{\alpha\beta}, \quad (\text{D.9})$$

$$Qv_{\alpha\beta} + \frac{1}{2}H^2(\Phi_{\alpha}^{-2\gamma}v_{\gamma\beta} - \Phi_{\beta}^{-2\gamma}v_{\gamma\alpha}) = iI_2\epsilon_{\alpha\beta}, \quad (\text{D.10})$$

$$H^2\Phi_{\alpha\alpha}\Phi_{\beta\beta}^{-1}v^{\alpha\beta}v^{\alpha\beta} = -4(I_2^2 + QX), \quad H^2\Phi_{\alpha\alpha}^{-1}\Phi_{\beta\beta}^{-1}v^{\alpha\beta}v^{\alpha\beta} = 4X \quad (\text{D.11})$$

$$\mathbf{e}^a \wedge \mathbf{e}^b = \frac{1}{8}\Gamma_{\alpha\beta}^{ab}\mathbf{E}^{\alpha\beta}, \quad \mathbf{E}_{\alpha\beta} = \Gamma_{\alpha\beta}^{ab}\mathbf{e}_a \wedge \mathbf{e}_b. \quad (\text{D.12})$$

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